Genetic Programming for Finite Algebras

— *GECCO-2008* Presentation —

Lee Spector *
David M. Clark †
Ian Lindsay *
Bradford Barr *
Jon Klein *

* Cognitive Science, Hampshire College, Amherst, MA
† Mathematics, SUNY New Paltz, New Paltz, NY

Outline

- The domain
- Specific problems
- Methods
- Results
- Significance

Everybody's Favorite Finite Algebra

$$\text{Boolean algebra, } \mathbf{B} := \langle \{0,1\}, \wedge, \vee, \neg \rangle$$

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

	$\neg$
0	1
1	0

Primal: every possible operation can be expressed by a term using only (and not even) $\wedge$, $\vee$, and $\neg$.

Bigger Finite Algebras

- Have applications in many areas of science, engineering, mathematics
- Can be *much* harder to analyze/understand
- Number of terms grows astronomically with size of underlying set
- Under active investigation for decades, with major advances (cited fully in the paper) in 1939, 1954, 1970, 1975, 1979, 1991, 2008

Goal

- Find terms that have certain special properties
- *Discriminator* terms, determine primality

$$t^A(x, y, z) = \begin{cases} x \text{ if } x \neq y \\ z \text{ if } x = y \end{cases}$$

- *Mal'cev*, *majority*, and *Pixley* terms
- For decades there was no way to produce these terms in general, short of exhaustive search
- Current best methods produce enormous terms

Specific Algebras

$\mathbf{A}_1$ *	0	1	2
0	2	1	2
1	1	0	0
2	0	0	1

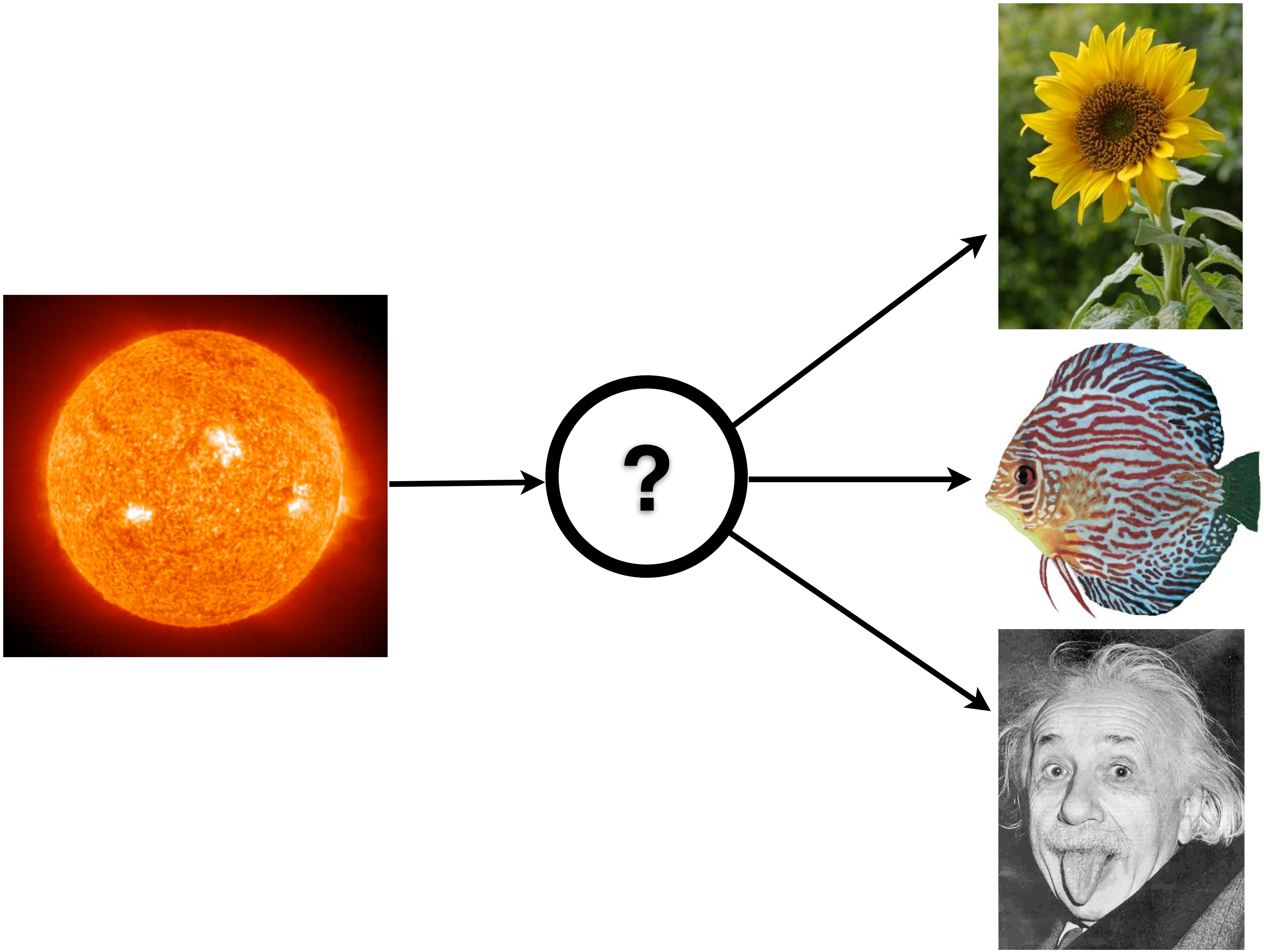
$\mathbf{A}_2$ *	0	1	2
0	2	0	2
1	1	0	2
2	1	2	1

$\mathbf{A}_3$ *	0	1	2
0	1	0	1
1	1	2	0
2	0	0	0

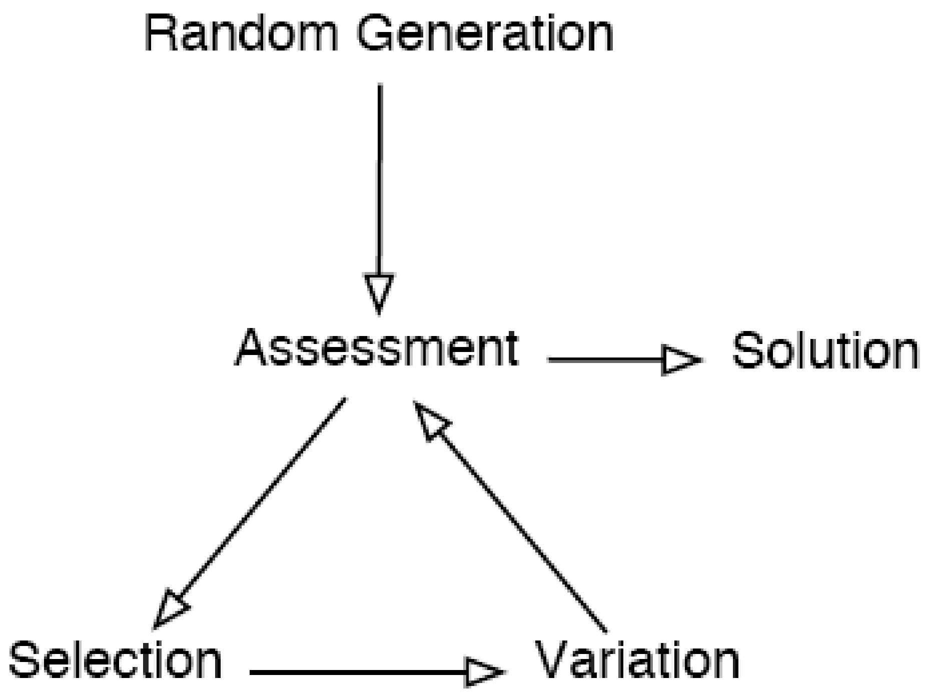
$\mathbf{A}_4$ *	0	1	2
0	1	0	1
1	0	2	0
2	0	1	0

$\mathbf{A}_5$ *	0	1	2
0	1	0	2
1	1	2	0
2	0	1	0

$\mathbf{B}_1$ *	0	1	2	3
0	1	3	1	0
1	3	2	0	1
2	0	1	3	1
3	1	0	2	0



Genetic Programming



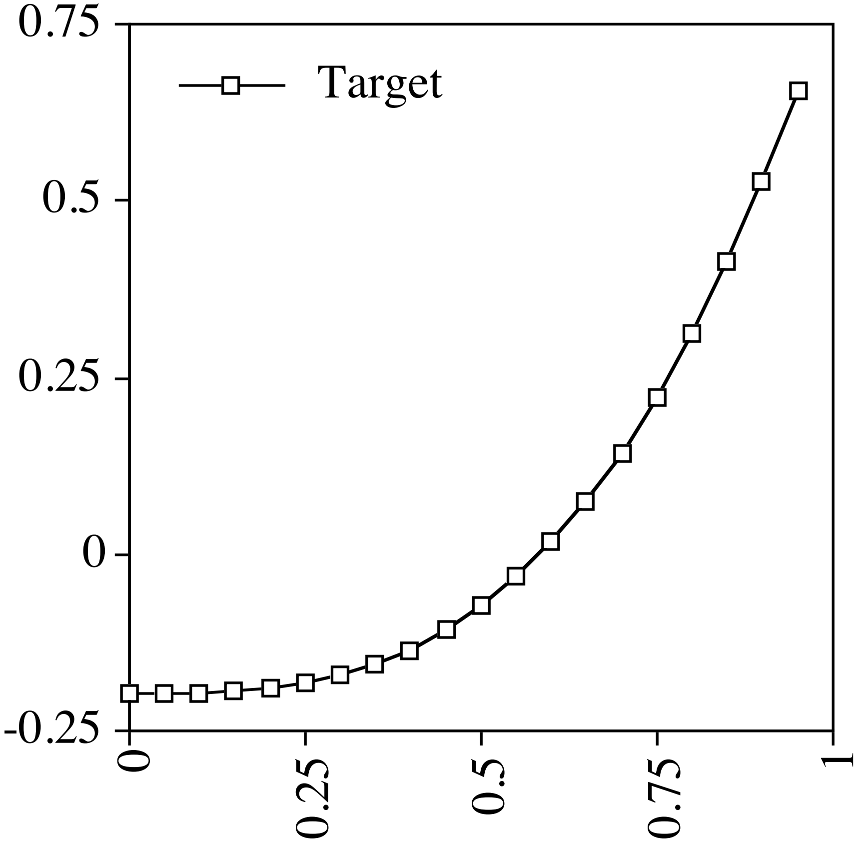
Numerical Example

Given a set of data points, evolve a program that produces *y* from *x*.

Primordial ooze: +, -, *, %, *x*, 0.1

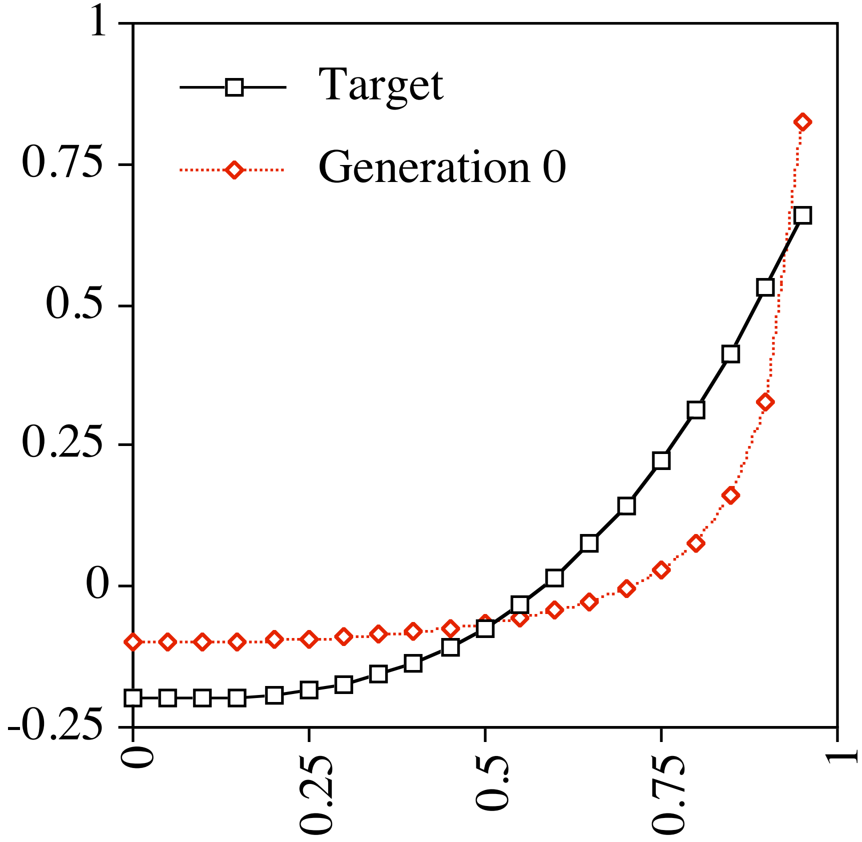
Fitness = error (smaller is better)

Evolving $y = x^3 - 0.2$



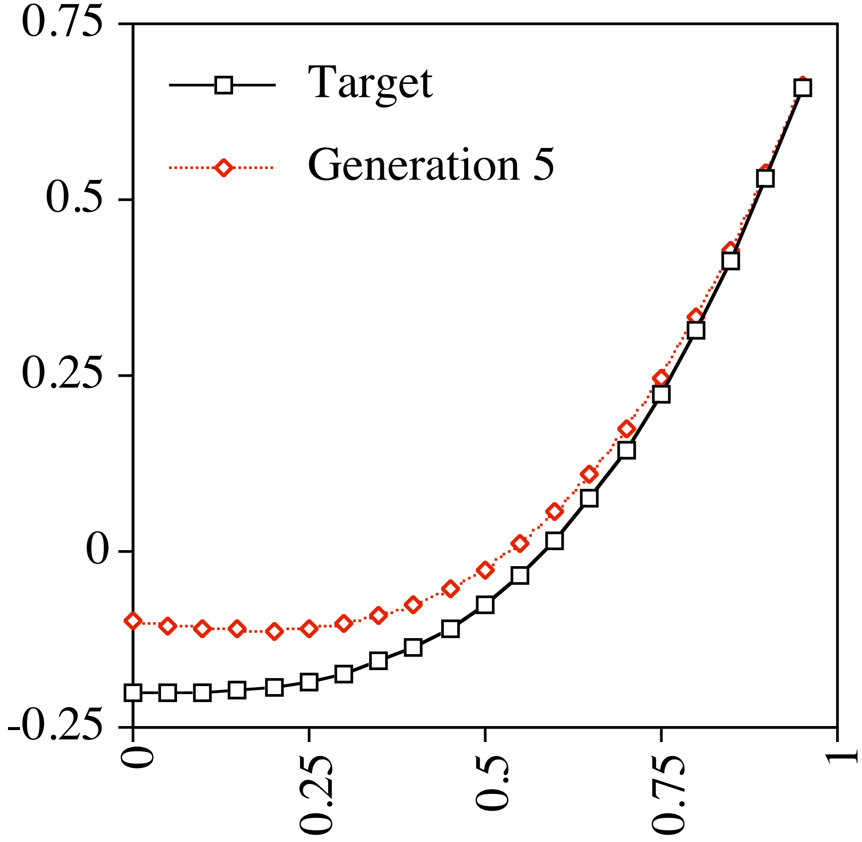
Best Program, Gen 0

```
(- (% (* 0.1
         (* X X))
      (- (% 0.1 0.1)
         (* X X)))
   0.1)
```



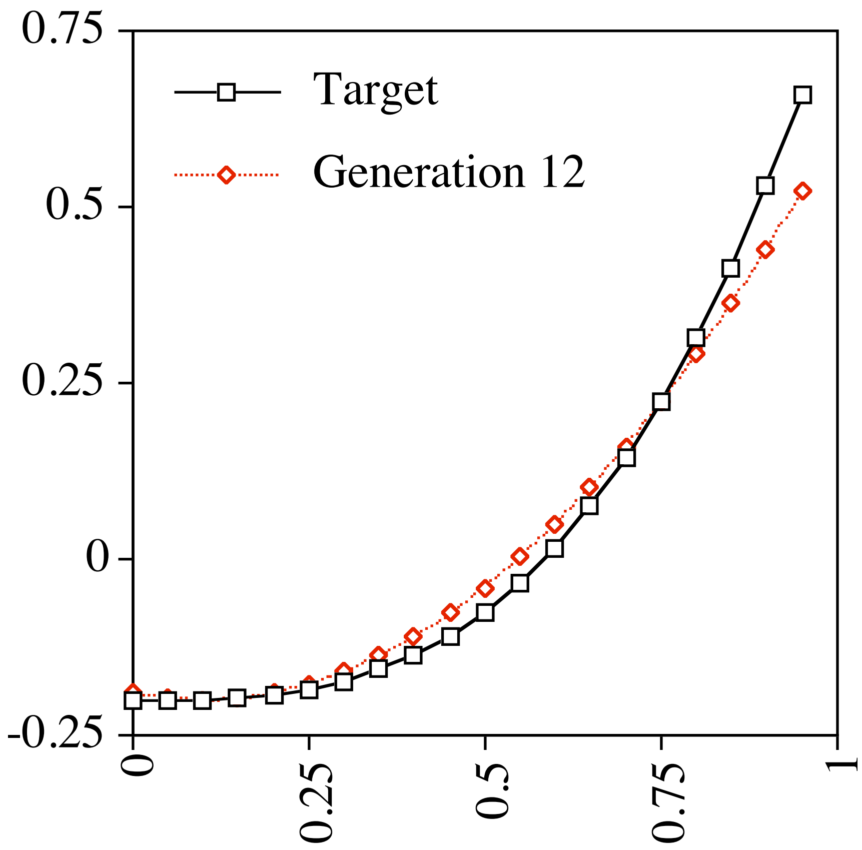
Best Program, Gen 5

```
(- (* (* (% X 0.1)
         (* 0.1 X))
      (- X
         (% 0.1 X)))
   0.1)
```



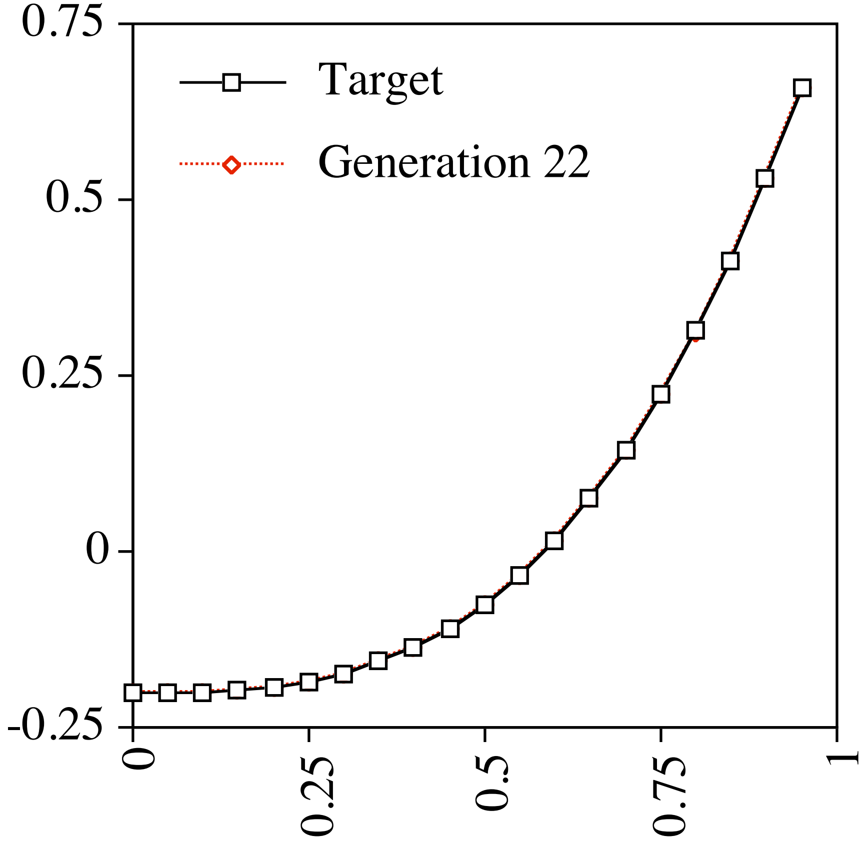
Best Program, Gen 12

```
(+ (- (- 0.1
         (- 0.1
            (- (* X X)
               (+ 0.1
                  (- 0.1
                     (* 0.1
                        0.1))))))
      (* X
         (* (% 0.1
               (% (* (* (- 0.1 0.1)
                        (+ X
                           (- 0.1 0.1)))
                     X)
                  (+ X (+ (- X 0.1)
                          (* X X)))))
            (+ 0.1 (+ 0.1 X))))))
   (* X X))
```



Best Program, Gen 22

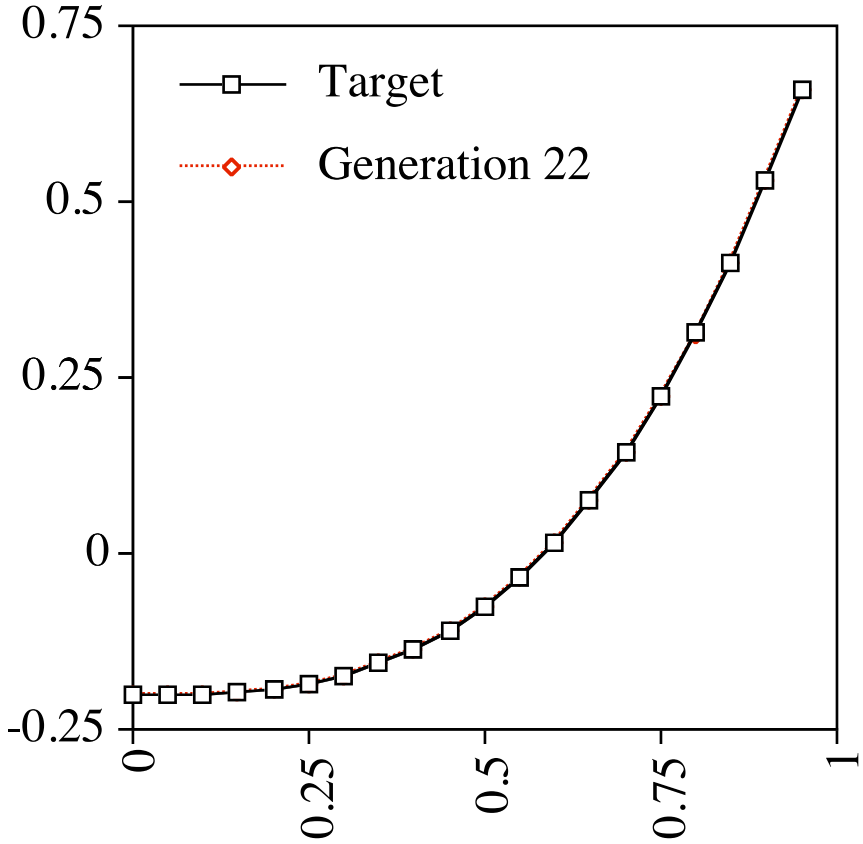
```
(- (- (* X (* X X))
      0.1)
   0.1)
```



Best Program, Gen 22

"... removal of any one of the parts causes the system to effectively cease functioning."

```
(- (- (* X (* X X))
      0.1)
   0.1)
```



Best Program, Gen 22

"... removal of any one of the parts causes the system to effectively cease functioning."

= "irreducably complex" (Behe)

= Evidence for an intelligent designer!

Produced by 100% Darwinian means!

(Reductio ad absurdum)

Evolution, the Designer

Apparent "irreducible complexity" is actually an expected product of Darwinian mechanisms, not evidence for a non-Darwinian "designer."

"Darwinian evolution is itself a designer worthy of significant respect, if not religious devotion." *Boston Globe* OpEd, Aug 29, 2005

WHAT WOULD DARWIN SAY? | LEE SPECTOR

The Boston Globe

And now, digital evolution

By Lee Spector | August 29, 2005

RECENT developments in computer science provide new perspective on "intelligent design," the view that life's complexity could only have arisen through the hand of an intelligent designer. These developments show that complex and useful designs can indeed emerge from random Darwinian processes.

Methods

- Traditional genetic programming with ECJ
- Stack-based genetic programming with PushGP
- Alternative random code generators
- Asynchronous islands
- Trivial geography
- Parsimony-based selection
- Alpha-inverted selection pressure
- HAH = Historically Assessed Hardness

Results

- Discriminators for A_1, A_2, A_3, A_4, A_5
- Mal'cev and majority terms for B_1
- Parameter tables and result terms in paper
- Example discriminator term for A_1:

$$((((((((x*(y*x))*x)*z)*(z*x))*((x*(z*(x*(z*y))))*z))*z)*z)*(z*((((x*(((z*z)*x)*(z*x)))*x)*y)*(((y*(z*(z*y)))*(((y*y)*x)*z))*(x*(((z*z)*x)*(z*(x*(z*y))))))))$$

Assessing Significance

Relative to prior methods:

- Uninformed search:
  - Exhaustive: analytical (expected value) and empirical search time comparisons
  - Random: analytical (expected value) and empirical search time comparisons
- Primality method: empirical term size comparisons

Expected Value Analysis

Since $\operatorname{Exp}(X)$ is the weighted sum of the values of X,

$$\operatorname{Exp}(X) = \sum_{j=1}^{\infty} j p_j = \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} p_j = \sum_{k=1}^{\infty} P_k \approx \sum_{k=1}^{\infty} \left(\tfrac{n-1}{n}\right)^{k-1}$$

$$= \frac{1}{1 - \frac{n-1}{n}} = n.$$

We recapitulate this conclusion as follows.

The expected value $\operatorname{Exp}(X)$ of the number X of trials required to find a term representing the function f is approximately the size $n = |A|^{|B|}$ of the search space A^B of all functions from B to A.

- Verified via empirical results with random search and exhaustive search

Significance, Time

	Uninformed Search Expected Time (Trials)
3 element algebras	
Mal'cev	5 seconds ($3^{15} \approx 10^7$)
Pixley/majority	1 hour ($3^{21} \approx 10^{10}$)
discriminator	1 month ($3^{27} \approx 10^{13}$)
4 element algebras	
Mal'cev	10^3 years ($4^{28} \approx 10^{17}$)
Pixley/majority	10^{10} years ($4^{40} \approx 10^{24}$)
discriminator	10^{24} years ($4^{64} \approx 10^{38}$)

Significance, Time

	Uninformed Search Expected Time (Trials)	GP Time
3 element algebras		
Mal'cev	5 seconds ($3^{15} \approx 10^7$)	1 minute
Pixley/majority	1 hour ($3^{21} \approx 10^{10}$)	3 minutes
discriminator	1 month ($3^{27} \approx 10^{13}$)	5 minutes
4 element algebras		
Mal'cev	10^3 years ($4^{28} \approx 10^{17}$)	30 minutes
Pixley/majority	10^{10} years ($4^{40} \approx 10^{24}$)	2 hours
discriminator	10^{24} years ($4^{64} \approx 10^{38}$)	?

Significance, Size

Term Type	Primality Theorem
Mal'cev	10, 060, 219
Majority	6, 847, 499
Pixley	1, 257, 556, 499
Discriminator	12, 575, 109

(for A_I)

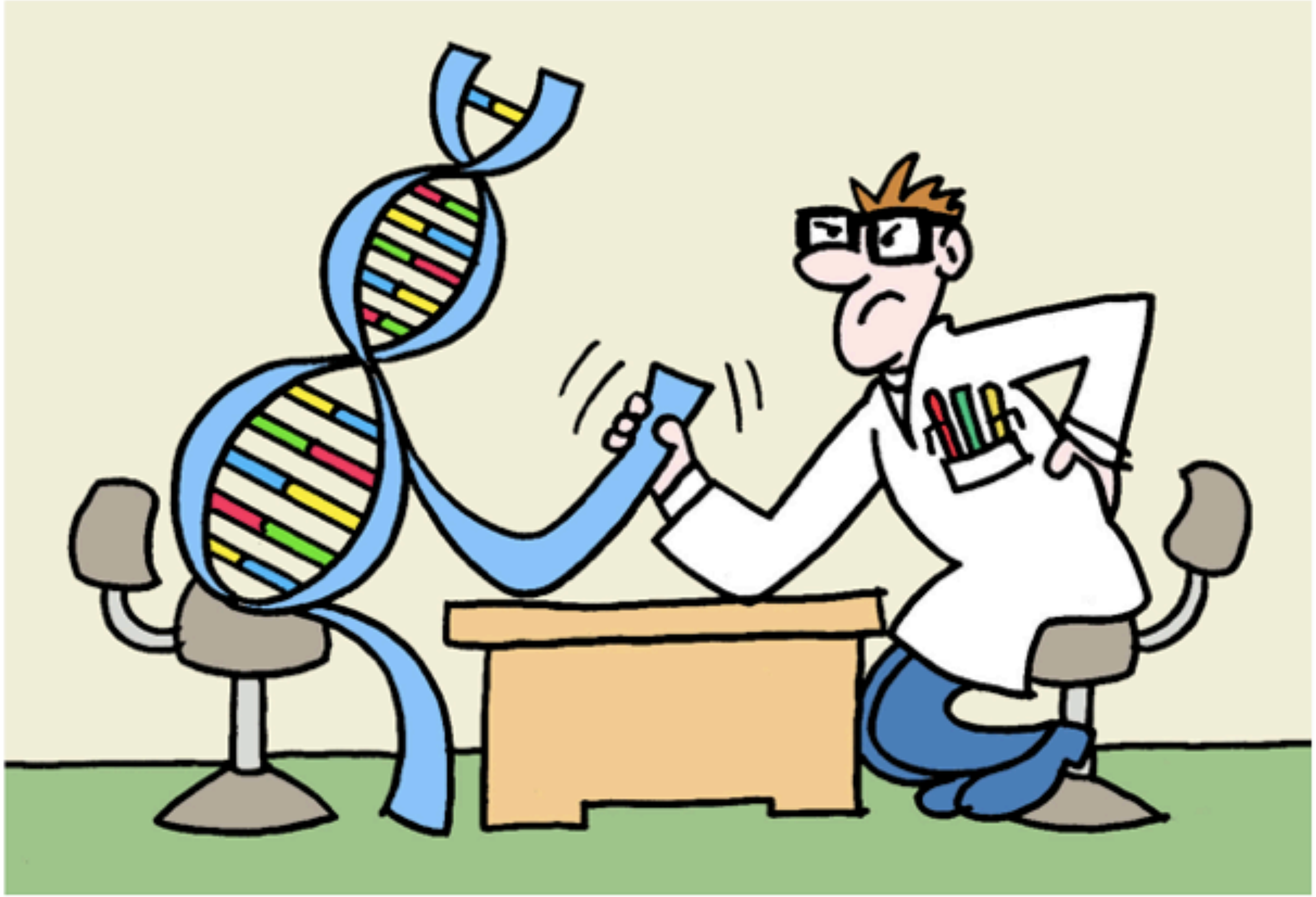
Significance, Size

Term Type	Primality Theorem	GP
Mal'cev	10, 060, 219	12
Majority	6, 847, 499	49
Pixley	1, 257, 556, 499	59
Discriminator	12, 575, 109	39

(for A_1)

http://www.human-competitive.org/

THE 5th ANNUAL (2008) "HUMIES" AWARDS FOR HUMAN-COMPETITIVE RESULTS PRODUCED BY GENETIC AND EVOLUTIONARY COMPUTATION HELD AT THE GENETIC AND EVOLUTIONARY COMPUTATION CONFERENCE



Human Competitive?

- Rather: human-**WHOMPING!**
- *Outperforms* humans *and all other known methods* on significant problems, providing benefits of *several orders of magnitude* with respect to search speed and result size
- Because there were no prior methods for generating practical terms in practical amounts of time, GP has provided the first solution to a previously open problem in the field

Potential Impact

These results are in an foundational area of pure mathematics with:

- A long history
- Many outstanding problems of theoretical significance and quantifiable difficulty
- Applications across the sciences

Conclusions

- Using GP, we have improved significantly on extensive past efforts of both humans and machines to solve problems related to finite algebras
- This is an important and previously unexplored application area for GP, with many open problems and quantitative measures of success

Genetic Programming for Finite Algebras

Lee Spector
Cognitive Science
Hampshire College
Amherst, MA 01002
lspector@hampshire.edu

David M. Clark
Mathematics
SUNY New Paltz
New Paltz, NY 12561
clarkd@newpaltz.edu

Ian Lindsay
Hampshire College
Amherst, MA 01002
iml04@hampshire.edu

Bradford Barr
Hampshire College
Amherst, MA 01002
bradford.barr@gmail.com

Jon Klein
Hampshire College
Amherst, MA 01002
jk@artificial.com

Humies 2008

GOLD MEDAL!